Mathematical modelling as a tool for the connection of school mathematics

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Abstract: *We start introducing some aspects of the theoretical framework: the Anthropological Theory of Didactics (ATD). Then, we consider on the research domain commonly known as "modelling and applications" and briefly describe its evolution using the ATD as an analytical tool. We propose a reformulation of the modelling processes from the point of view of the ATD, which is useful to identify new educational phenomena and to propose and tackle new research problems. Finally, we focus on the problem of the connection of school mathematics. The reformulation of the modelling processes emerges as a didactic tool to tackle this research problem. We work on the problem of the articulation of the study of functional relationships in Secondary Education and present a teaching proposal designed to reduce the disconnection in the study of functional relationships in Spanish Secondary Education.*

ZDM-Classification: D20, D30, F80, I24, M14

1. Introducing the Anthropological Theory of the Didactic

The works of Chevallard (1999, 2006), Chevallard, Bosch, Gascón (1997), Gascón (2003), Barbé, Bosch, Espinoza and Bosch, Chevallard and Gascón (2006) show how the Anthropological Theory of the Didactic (from now on ATD) has emerged and developed within the research in Didactics of Mathematics. It proposes, as we will see, a new way of modelling the mathematical activity and its teaching and learning through the notions of mathematical and didactic *praxeologies*. It also introduces a scale of *levels of mathematical and didactic determination* to study the generic restrictions coming from society, school or the different disciplines taught at school, as well as the more specific ones coming from the way school mathematics is organised and divided into "blocks of contents", domains, themes and topics. After giving a short account of the ATD approach, we will see how to consider mathematical modelling and how to formulate and approach what we call the problem of "connecting the mathematical curriculum".

1.1. The modelling of mathematical activity: mathematical *praxeologies*

The ATD proposes a general epistemological model of mathematical knowledge, conceived as a human activity. The main theoretical tool is the notion of *praxeology* (or *mathematical organization*) that is structured in two levels:

- The *praxis* or "know how", which includes different *kinds of problems* to be studied as well as *techniques* available to solve them.
- The *logos* or "knowledge", which includes the "discourses" that describe, explain and justify the techniques used and even produce new techniques. This is called *technology* in its etymological sense of "discourse (logos) on the technique (technè)". The formal argument which justifies such technology is called theory. It is conceived as a second level of description-explanation-justification.

The ATD assumes an institutional conception of the mathematical activity. Mathematics, like any other human activity, is something that is produced, taught, learned, practised and diffused in social institutions. It can be modelled in terms of praxeologies called *mathematical praxeologies* or *mathematical organizations* (MO from now on). For instance, in Spanish Secondary Education, it is possible to identify a mathematical praxeology around proportional relationships including a set of problematic tasks (the classic proportional problems where three measures are given and a fourth one is to be found), techniques to deal with these problems (commonly known as *rule of three*) and a technological-theoretical discourse that explains and justifies the mathematical activity performed (defining what are proportional magnitudes and how to determine if two magnitudes are directly or inversely proportional). But this MO is mixed up with praxeological components from other praxeologies: on the one hand, the praxeology that considers proportionality as a relationship between numerical variables modelled by an equation $y = k \cdot x$; on the other hand, the praxeology that considers proportional relationships as linear functions (see García, 2005, for more details).

In order to have the most precise tools to analyse the institutional didactic processes, Chevallard (1999, p. 226) classifies mathematical praxeologies as *specific, local, regional* and *global*. The nature of a praxeology depends on the institution where it is considered:

- A *specific praxeology* is generated by a unique type of task. Generally, in a specific praxeology there is only one technique to deal with the task and represents the "official" way to solve the problem in that institution. The technology is usually absent or implicitly assumed.
- A *local praxeology* is generated by the integration and connection of several specific praxeologies. A local praxeology is characterized by a technology that justifies, explains, connects and produces the different techniques of each specific praxeology.
- A *regional praxeology* is obtained as the result of the coordination, integration and articulation of several local praxeologies in a common mathematical theory.
- *A global praxeology* emerges when several regional praxeologies are added together (integration or juxtaposition of different mathematical theories).

Thus, the classic praxeology around proportional relationships integrates different specific praxeologies (direct rule of three, inverse rule of three and multiple rule of three) with a common technology: the theory of ratios and proportions. This local praxeology is in turn an element of a regional one which includes fractions, measures and several applications of ratios and proportions to commercial arithmetic.

In short, what is learned and taught in an educational institution are mathematical praxeologies. In general, praxeologies are shared by groups of human beings organized in institutions. *Cognition* is thus institutionally conceived.

1.2. The study process: didactic praxeologies

Mathematical praxeologies do not emerge suddenly in an institution. They do not have a definite shape. On the contrary, they are the result of a complex and ongoing activity, where some invariable relationships, which can be modelled, exist. There appear two indivisible aspects of the mathematical activity: on the one hand, the process of mathematical construction (the *study process* or *didactic process*) and, on the other hand, the result of this construction (the *mathematical praxeology*). In fact, there is no *mathematical praxeology* without a *study process* that engenders it but, at the same time, there is no *study process* without a mathematical praxeology in construction. *Process* and *product* are the two sides of the same coin.

> "Generally speaking, mathematical activity can be considered as the use of a mathematical organization or a mathematical work. However it is also, at the same time, a production (or re-production) of mathematical realities that will lead to new mathematical organizations. The English term "work" (translated from the French oeuvre) allows us to talk about mathematics as a human activity –given that mathematics are something we do- and as an artefact produced and reproduced by this activity –the work of mathematicians-. A mathematical work is something to be used and something to be produced or reproduced." (Bolea, Bosch & Gascón 1999, p. 136).

The consideration of different processes of mathematical construction shows some invariable dimensions or *moments* that structure them, independently of cultural, social or individual factors. The *didactic moments* are defined, not in a chronological or linear sense, but as different dimensions of the mathematical activity.

Thus, the *study process* can be situated in a space characterized by six *didactic moments*: (1) *the moment of the first encounter* with a specific type of tasks, (2) *the moment of the exploration* of the type of tasks, (3) *the moment of the construction of the technological-theoretical environment* (that explains and justifies the techniques used and will also allow the production of new techniques), (4) *the moment of working on the technique* (which provokes the evolution of the existing techniques and the creation of new ones), (5) *the moment of institutionalization* (where the components of the praxeology constructed are delimited) and (6) *the moment of evaluation* of the praxeology constructed.

The study process does not have a linear structure. Each *moment* can be lived with varying intensity at different moments during the study process as many times as necessary. It is common that some of them occur simultaneously. Each *moment* has a specific function in the development of the study process and there exists a global internal dynamic manifested in the invariable character of certain relations among these *moments*.

A study process (like every human activity) can be modelled in terms of praxeologies which are now called *didactic praxeologies* (Chevallard 1999, p. 244). Like every *praxeology*, didactic praxeologies include a set of problematic educational tasks, educational techniques (to tackle these tasks) and educational technologies and theories (to describe and explain these techniques).

For instance, the introduction of a new concept, like the concept of *proportional relationships* in Secondary Education, is a problematic educational task. There is not a unique way to perform this task. In many cases, this task is explicitly assumed by the teacher through the production of a discourse followed by some examples. This educational technique is justified by a specific representation of the didactic system and the way that students construct their mathematical knowledge (that can be summarized in the following slogan: students learn what the teacher explains clearly). However in other cases this problematic educational task is "shared" by the teacher and the students. A way to do that could be the investigation of the variation in a specific real situation (for instance, the dimensions of a shadow projected on a screen when the distance between the torch and the object varies[1]). This educational technique is now justified by another representation of the didactic system and the way that students construct their mathematical knowledge (that can be summarized in the slogan taken from "Hot Math!" website: *I hear, I forget. I see, I learn. I do and I understand!*).

[1] Taken from Hot Math! Website (http://projects.edte.utwente.nl/hotmath/index.html)

A new conception of didactics of mathematics arises. *Didactic* is *identified* with anything that can be related to *study* and *helping to study*:

> "Didactics of mathematics is the science of study and helping to study mathematics. Its aim is to describe and characterize the study processes (or didactic processes) in order to provide explanations and solid answers to the difficulties which people (students, teachers, parents, professionals, etc.) face when they are studying or helping others to study mathematics" (Chevallard, Bosch y Gascón 1997, p. 60).

1.3. The levels of determination

The mathematical knowledge is produced, taught, learned, practised and diffused in social institutions. It is thus not possible to separate it from its process of construction in a specific institution. Chevallard (2001, 2002a, 2002b) proposes a hierarchy of determination levels among mathematical praxeologies that live (or could live) in an institution and the possible ways of constructing those praxeologies in this institution (the *didactic praxeologies*):

Civilization → Society → School → Pedagogy → Discipline → Domain → Sector → Theme → Subject

The structure of the praxeologies on each level of the hierarchy conditions the possible ways of organizing its study, that is, the didactic praxeologies. Reciprocally, the nature and the function of the *didactic tools* existing in each level determine, to a large extent, the type of praxeologies that could be reconstructed.

Every question Q that generates a didactic process in an educational institution is embedded in a theme belonging to a sector included in a domain of a discipline. If the discipline is *mathematics*, we will refer to these levels as *mathematical levels*. In contrast, the levels beyond the "discipline" are considered as *pedagogical levels*.

> "For instance, the question "Which are the symmetries of a rectangle (not squared)?" is considered, in most educational systems, as belonging to the theme "symmetries of polygons", which is included in the sector "transformations", included in the domain "Geometry", belonging to the discipline "Mathematics"" (Chevallard, 2001, p. 3).

However, the construction of that hierarchy does not guarantee the quality of the study of Q. For a question Q to be studied meaningfully at school, it is also necessary that (1) the question Q comes from those questions that society proposes to be studied at school (*cultural* or *social legitimacy*), (2) Q appears in certain "*umbilical"* mathematical *situations*, that is, it is placed at the *core* of mathematics (*mathematical legitimacy*) and (3) Q *leads us somewhere*, that is, it is connected to other questions studied at school, either mathematical questions or questions from other disciplines (*functional legitimacy*).

If a hierarchy taking into consideration (1), (2), (3) is not constructed for a specific question Q, there is no point in studying it because the question has lost its rationale or *raison d'être*. In that case it is said that Q is a *dead question* (Chevallard, Bosch and Gascón, 1997).

In García (2005), for example, we showed that in Spanish Secondary Education there exist two different hierarchies related to the study of the proportional relationship between magnitudes. The first one places the proportional relationship in the sector of "Proportionality" and in the domain of "Numbers and measures". This implies that proportionality is conceived as a static relationship modelled in terms of *proportions*. The second hierarchy places the proportional relationship in the sector of "Characterizing relationships between magnitudes", which is a part of the domain called "Functions and their graphical representation". This implies that proportionality is conceived as a dynamic relationship modelled in terms of *linear functions*. The existence of the two hierarchies gives rise to the reconstruction of two different praxeologies in present Secondary Education, studied at different moments of the school year and almost completely disconnected.

Traditionally, the work of the teacher has been limited to the "Theme → Subject" levels, leaving the higher levels to be determined by the official curriculum and educational authorities. This phenomenon, identified by Chevallard (2001) as the "*phenomenon of the teacher's confinement*" in the theme-subject levels, does not help mathematical themes and questions studied at school to explicitly show the reasons that motivated their presence in the curriculum, because these reasons are usually located at higher levels of determination, in the connection between different contents or praxeologies. The "thematic confinement" arrives when teachers do not question the way mathematical contents are organized into blocks, around important "generative questions" (like the use of plane geometry –triangles– to approach the problem of measurement, the link between proportionality and functional modelling, or the relation between derivatives and physical mechanics). In this case, the contents may turn into *dead questions*, when the institution seems to ignore where they come from and where they lead to. We can then talk about the *monumentalization of mathematical organizations* phenomenon: the students are invited to *visit* but not to *construct* them.

In some cases, research in mathematic education also seems to be restricted to the "thematic" level, focused in studying the "appropriate" way of introducing a mathematical content – a *specific praxeology* – in a given educational institution, without any deeper reflection upon the way this content is structured and without taking into account the conditions and restrictions imposed by the different co-determination levels during the didactic transposition process: why this praxeology belongs to this block and no to this other one, why does it appear in the curriculum, which is its rationale, its motivation, where does it come from, etc.

2. Research in Mathematical modelling

Since the mid-eighties, researchers in Mathematics Education have a growing interest in the role that modelling processes can play in the teaching and learning of mathematics in all levels of the educational system. For a long time, "modelling" has been restricted to the application of a mathematical knowledge, already constructed, to a specific "real" situation. Even if this use of the term persists, "modelling" is considered in a richer and more fertile way in Mathematics Education, where it forms a large research domain that is constantly growing.

At present, it is common to hear, both from the mathematical education community and from different social agents, about the necessity of linking the mathematical contents to certain aspects of real life and about the necessity of developing the "modelling competence" as a basic mathematical competence in students, as it is shown, for instance, in the recent PISA study (OECD, 2003).

Two different interpretations of "modelling" can be considered:

- On the one hand, the *idea* of taking, from mathematics, the processes of mathematical modelling as a "powerful didactic tool". In other words, the problem approached can be formulated as follows: *how could modelling processes improve the teaching of mathematics and the understanding of mathematical concepts*?
- On the other hand, modelling also refers to the necessity of an explicit teaching of modelling processes as a specific mathematical content, linked to the students' formative needs (for instance, biology, chemistry or engineering students). We can resume this trend in the following starting question: *how could students achieve a modelling competence in relation to their specific scientific or professional field?*

The work done in these initial questions causes a first development of the research domain called "modelling and applications", producing different trends not disconnected between them, although we will present them here separately.

A first trend concentrates on the search of "good" systems to be modelled and "good" applications of mathematics. Searching in the wide universe of systems and models, this trend tries to identify those systems "appropriate" to engage students in a modelling process that allows them to achieve the desired mathematical content or the "competence" to develop modelling processes on their own. There is no deep theoretical base explicitly constructed. The theoretical foundation is left in the hands of other disciplines, like pedagogy or psychology.

A second trend focuses on how to manage these "good" systems and models within the teaching and learning processes. From the beginning, the notion of "modelling" was taken from "pure mathematics" and summarized in the well-known "modelling cycle" (see, for instance, Blum & Niss 1991). Nowadays, this kind of research continues and produces, through reflection and experimentation, good examples of modelling and mathematical applications ready to be performed in the classroom.

However, the evolution of the research in "modelling and applications" led to a growing interest in the modelling process in itself. As Niss (1999) established, there is no automatic transfer from a solid knowledge of mathematical theory to the ability to solve non-routine mathematical problems. He proposes that "problem solving" and "modelling" have to be the object of teaching and learning. Consequently, "modelling" becomes an object of research.

The study of the modelling process has been carried out initially from two different perspectives, not necessarily independent:

- *An epistemological approach* focused on the characteristics of the "real situations" involved in the modelling processes used with didactic purposes, also including the question of the relation between those situations and the mathematical knowledge (questioning that these "real situations", on their own, have didactic properties).
- *A cognitive approach* that emerges from the necessity of a deeper understanding of the cognitive processes that students activate when they are involved in *modelling / application tasks*.

It is well known that the question of students' *cognitive processes* has provoked the emergence of new research trends that have produced important research results. However, this questioning has also provoked an inversion in the way of questioning the modelling process: as the *cognitive analysis* shows that performing *modelling and applications tasks* involve the activation of complex cognitive processes, the necessity of an explicit teaching of *modelling techniques and skills* in educational institutions (not necessarily specialized) emerges. Thus, the two initial approaches converge in what can be summarized as the "problem of modelling", formulated as follows:

How to get students develop non routine modelling skills by themselves (generally referring to extra-mathematical situations)?

Although there exist a great variety of researches dealing with this topic what makes any attempt to organise it difficult (see Blum, 2002), we propose a possible structure using the *determination levels* explained above.

A great part of the existing research places "modelling" on a *thematic level*, giving rise to the construction of specific and isolated mathematical praxeologies. In general, the research problem can be synthesized as follows: *how does a person act when s/he is trying to solve a specific "real problem" that implies the necessity to model a system?* It is specific because the research problem is confined to what the institutions consider a unique type of task. It is generally limited to a particular modelling process, that is, to an isolated system (generally extra-mathematical). The student has to construct a model that represents this system, work within the model and obtain a solution that needs to be confronted with the initial system. Once the model has been constructed and the solution has been found, the system disappears, the *model* becomes part of the student's *mathematical heritage* and a new modelling process begins, not necessarily connected to the previous one. The isolated nature of that type of problems often makes them become anecdotic: the student ignores where they come from and where they lead to. All happens as if the modelling process was explicitly designed for the student to construct a specific concept or to put it to the test. Once this objective has been achieved, the system and the question that initiates the modelling process "dies" and "disappears". Modelling is here a means to reach the goal of constructing new knowledge. At present there are a large number of researches dealing with the "problem of modelling" at this *specific level*, as shown, for instance, in the different editions of the conference of the International Community of Teachers of Mathematical Modelling and Applications (ICTMA).

Apart from researches confined to the *specific level*, there exists another trend where the "problem of modelling" is situated at a more generic level: the one of the *discipline* or even the *links between disciplines*. In a general way, the problematic question can be summarized as follows: *how to get students develop general modelling skills or a general modelling competence* linked to the *problem solving* competence? This problem is located at the *discipline level* because it is formulated independently of any mathematical content and without taking into account the different structuring levels of mathematical knowledge (domains, sectors, themes). Although the modelling competence is developed through particular modelling examples, these are only a means for the general purpose: the development of this general competence. There exits again a great variety of points of view in this general trend: for example, the research focused in the students'/teachers' beliefs when facing modelling tasks (Maaβ, 2005).

A consequence of the formulation of the "problem of modelling" at the *discipline level* is that one of the central aspects for research in "modelling and applications" should be the curriculum development. Then terms like *mixed curricula* or *integrated curricula* emerge, referred to curricula where modelling questions appear either as another theme or mixed with the mathematical themes. We could refer for instance to the Danish project KOM (Niss, 2003) where the general aims of mathematical education are reformulated in terms of competences development, including the *modelling competence* as a general competence that students should develop at school.

Some research trends try to place the "problem of modelling" at the *sector level*, that is at an intermediate level between the themes and the whole mathematics as a discipline. For instance, the research domain known as Realistic Mathematics Education[2], that considers mathematics as a human activity, proposes to build *didactic trajectories* that aim to construct mathematical knowledge starting from "real situations"[3] through the process of horizontal/vertical mathematization (from a "model of" to a "model for").

Most of the educational research carried out in the domain of "modelling and applications" coincides in using the "modelling cycle" in the description of the modelling processes, with very few variations. At the most, there is some questioning of the cognitive processes activated in each step of the modelling cycle or in the transition between different steps. This has led to an enriched version of the modelling cycle

[2] Freudenthal (1973, 1991), Treffers and Goffree (1985), De Lange (1996)
[3] The term "real" is not conceived in the "real life" sense, but in the sense of "real for students".

(for instance, the one proposed in Blum, 2006). In other words, the notion of "modelling" is not considered as a problematic notion in the research carried out in mathematics education (as it is not problematic in biology or engineering, where it is also used). What is problematic is the teaching and learning of modelling or the use of modelling for the teaching and learning of mathematics. In fact, the built "patterns" of the *modelling processes* are very close to those suggested by mathematics itself. They are seldom modified or extended from the considered experimental facts.

It is undeniable that a great progress has been made and that important new results are expected in the future. However, it is also true that nowadays the results obtained are far from the desired ones. As Blum (2002) says:

> "While applications and modelling also play a more important role in most countries' *classrooms* than in the past, there still exists a substantial gap between the ideals of educational debate and innovative curricula, on the one hand, and everyday teaching practice on the other hand." (p. 150)

It is necessary, then, to continue working on the study of the modelling processes and on their relevance to the teaching and learning of mathematics, both from the cognitive and the epistemological dimensions. Here we will focus on the epistemological and institutional dimensions. A way to do that, not developed enough in the existing research, could be through the general epistemological framework of mathematics proposed by the ATD, trying to reformulate the modelling processes within this general theoretical framework. To progress, we will work on a new reformulation of the modelling processes, beyond the "modelling cycle", as a tool to identify new educational problems and find possible solutions. Our aim is not to criticize the "modelling cycle" but to "transpose" it[4] into a solid theoretical framework.

3. Reformulating mathematical modelling in the ATD

The research paradigm known as "didactics of mathematics" or "epistemological approach of didactics", originated in Guy Brousseau's first works of the 70s (Brousseau, 1997), places the question of the epistemological model of mathematics in the core of educational research. The main hypothesis can be summarized as follows: each *didactic phenomenon* has an essential mathematical component and, reciprocally, each *mathematical phenomenon* has an essential didactic component (Chevallard, Bosch & Gascón, 1997). From this new point of view, the "didactic facts" and the "mathematical facts" are inseparable. This idea inaugurates a new way to tackle the research in the didactics of mathematics through the questioning of the epistemological models of mathematical knowledge used in teaching institutions, including those used in educational research.

The Anthropological Theory of Didactics is placed in the Epistemological Programme and postulates that "most of the mathematical activity can be identified (…) with *a mathematical modelling activity*" (Chevallard, Bosch and Gascón, 1997, p. 51). This does not mean that *modelling* is just one more aspect or dimension of mathematics, but that mathematical activity is essentially a *modelling activity* in itself.

First, this statement is meaningful if the idea of *modelling* is not limited only to "mathematization" of non-mathematical issues, that is, when the intra-mathematical modelling is considered as an essential and inseparable aspect of mathematics. The algebraic modelling of geometry, or the geometrical representation of algebraic and arithmetical expressions are examples of intra-mathematical modelling:

> In Spanish Compulsory Secondary Education there are several moments where geometric notions act as models of algebraic notions. For instance, the graphic resolution of systems with two linear equations and two variables is modelled by straight lines that justify two possible cases: a unique solution if the lines intersect (the solution is the coordinates of the intersection point) or no solution if these lines are parallel. The solution of a linear inequality with two variables is also modelled using the graphic representation of the associated linear

[4] The term "transpose" is used here as a metaphor of the *didactic transposition* processes.

equation. And there are also algebraic notions that model geometric properties as, for instance, the metric relation between the sides of a rectangular triangle modelled by the algebraic expression $a^2 = b^2 + c^2$ (being a the hypotenuse and b, c the other two sides). Given two sides, this algebraic model produces knowledge about the triangle (a quadratic equation leads us to the unknown side). Furthermore, this algebraic model can produce knowledge about the nature of a given triangle: being a the largest side, if $a^2 < b^2 + c^2$ the triangle has three acute angles while if $a^2 > b^2 + c^2$ then the triangle has one obtuse angle and two acute angles.

Second, the axiom of mathematics being essentially a modelling activity can only be meaningful if a precise meaning is given to the *modelling activity.* In the framework of the ATD, what is relevant is not the specific problem situation proposed to be solved (except in "*life or death" situations*), but what can be done with the solution obtained –that is, with the constructed praxeology–. The only interesting problems are those that can be reproduced and developed into wider and more complex types of problems. The study of those *fertile problems* provokes the necessity of building new techniques and new technologies to explain these techniques. In other words, the research should focus on those *crucial questions* that can give rise to a rich and wide set of mathematical organizations. Sometimes, those *crucial questions* have an extra-mathematical origin, sometimes they have not.

We assume that the important starting point to design a *study process* should not be the realness of the situations or the initial questions, but the possibility they offer to create a set of well connected and integrated praxeologies that would allow the development of a wide mathematical activity in a teaching institution, taking into account the restrictions and conditions imposed by that institution.

Following the example explained before (algebraic notions as models of geometric notions and vice versa), the institution seems to ignore the reciprocal modelling relation between algebra and geometry. A consequence of this situation is that students use the same mathematical knowledge with different meanings and different functions ignoring the rich relations existing between them. This fact gives rise to a difficult research problem: in an institution like Compulsory Secondary Education, is it possible to formulate a set of *crucial questions* that will provoke articulated study processes where algebraic and geometric notions emerge one from the other? Which could be these *crucial questions*? To what extent do the restrictions imposed from the different determination levels over the institution condition these *crucial questions* and the possible study processes that could emerge from them?

We propose to reformulate the *modelling process* as a process of reconstruction and interconnection of praxeologies of increasing complexity (*specific* → *local* → *regional*). This process should emerge from an initial question that constitutes the rationale of the sequence of praxeologies. From this questioning, some *crucial questions* to be answered by the *community of study* should arise. The answers produced in this *study process* should then be materialized in a *regional praxeology.*

From this point of view:

- The notions of *model* and *systems* are widened to be considered as *specific* or *local* praxeologies.
- The *modelling processes,* which are normally described in terms of system-model relations and the *modelling cycle,* can be characterized in terms of praxeologies and relations between praxeologies.
- A praxeology can be considered as a *model* or a *system* depending on the kind of questions put; being a model of a system is a *function* of a praxeology, it is not in its nature.
- *Modelling process,* either as an *object* to be taught or as a means for the *teaching* and *learning* of particular mathematical contents, cannot be considered as independent from the rest of the mathematical activity.

This new interpretation of *modelling* entails a rupture with the tradition of "modelling and applications" developed over the last 25-30 years by the Mathematics Education community. However we consider that the reformulation explained above does not contradict the previously obtained results. On the contrary, in a certain sense, it complements them introducing a new didactic tool to structure the *modelling processes* and to integrate them in a more general epistemological model of institutional mathematical activities. We are convinced that the introduction of a deeper epistemological dimension (with a solid theoretical base) in the international debate on modelling can lead us to improve our future research.

4. The school mathematics disconnection: A research problem

At present, in many countries (particularly in Spain) the curriculum of the educational institutions is structured in three main sections: *concepts* or *conceptual contents, procedures* and *attitudes*. Each section is specifically expressed in a list of contents, generally not too structured.

Furthermore, the mathematical curriculum is structured in a set of *domains* or "block of contents" subdivided in different *sectors*. For instance, in Compulsory Secondary Education in Andalusia (one of the seventeen regions that form Spain) there are five domains called: *numbers and measures*, *algebra*, *geometry, functions and their graphic representation, statistics and chance*. The *geometry domain*, for instance, is structured in different *sectors*: *elements and organization of the plane*, *element and organization of the space* or *translations, symmetries and rotations in the plane*, etc.

It is assumed that all these contents form a bigger organization –called "mathematics"– but it does not establish how these contents should be connected (or separated), apart from some general considerations. In general, according to the syllabi instructions, *the problem solving* and *the applications of mathematics in "real" context* are supposed to have a "connecting power" of the different contents included in the various sectors and domains. For instance, we can read some proposals as:

> "The contents are organized in five nuclei (…). In each nucleus the different contents are formulated in an integrated way: specific procedures, forms of expression and particular representations, concepts, facts, habits and attitudes. Daily life situations or problems where the contents appear are also appropriate.
>
> The teaching and learning process has to integrate contents from different mathematical fields (as simultaneous or complementary). Starting from the same experiences, problematic situations or activities, knowledge related to magnitudes, arithmetic, geometry, algebra, statistics or probability can be jointly elaborated." (CECJA, 2002, pp. 146-147)

Thus, the general task of school but mainly the teacher's task can be described as follows: given a list of contents organised in different blocks, how to make students carry out some "real" mathematical activity to make sure they *enter* in the desired *mathematical works*[5]. The crucial problem is how to structure and organize the set of contents included in the curriculum, that is, the elaboration of a students' *programme of study*. We call this problem the *problem of the curriculum elaboration*, which can no be tackled independently of the *domains* and *sectors* the curriculum is structured in (imposed by the *legal sphere*) and of the restrictions proceeding from school (for instance, the distribution of the mathematic classes in three hours per week but in different days or the fact that every mathematical knowledge has to be evaluated in a relatively short period of time).

Usually it is assumed that the *problem of the curriculum elaboration* can be solved focusing on the *teaching*, independently of the mathematical knowledge that is the object of this teaching. That is, what it is supposed to be problematic is:

- The way contents should be selected, ordered and sequenced.
- The way to teach these contents, conceiving teaching as the set of actions teachers should carry out to ensure the students' learning.

However, without an explicit questioning of the mathematical knowledge involved, one ends up assuming that there is a relatively *universal* and *non problematic* way to describe and organize school mathematics. And the same happens with the distribution in *domains* and *sectors* provided by the curricular documents. Moreover, inasmuch as this curricular organization is imposed by the *legal sphere* (politicians, educational authorities, etc.), the teacher's role is to select the *questions* proposed to be studied at school and the *themes* in which these

[5] Oeuvres, in French

questions are included[6] (phenomenon of *teacher's confinement*). That will partly explain the interest of certain domains of mathematical education (close to teachers and to the daily practice in the classroom) in the design and testing of modelling processes and applications of mathematics to real situations. The aim is to produce readymade examples to implement in the classroom.[7]

The *epistemological programme of research in didactics of mathematics* tries to *fight against* this situation (obviously in a metaphorical sense) through the introduction of the epistemological model of mathematics questioning, instead of considering it transparent and fixed.

> "The didactic point of view claims that the *problem of the curriculum elaboration* (…) has an *essential mathematical component*. The problem is not only to order and sequence the curriculum contents (…). It is a full *creative reconstruction* of the works that form the curriculum." (Chevallard, Bosch and Gascón, 1997, p. 127)

The *problem of the connection of school mathematics* could be initially described as follows:

> How to organize the school teaching of mathematics so as to provoke the connection of the different types of contents: concepts, procedures and attitudes? How to obtain that the *mathematical knowledge learned by students* will not be reduced to a set of disconnected techniques more or less algorithmic and without any sense?

However, this first formulation, which rises from a first observation of the didactic system, needs to be reformulated into a research problem. That is, the assumption of a theoretical framework not only determines the possible answers that can be produced but also the possible research problems to work on. As a matter of fact, it is formulated more like a problem of the individuals of the institution (teacher and students) than like a problem of the *mathematics teaching system*, implicitly assuming that there exists a *unique* and *unquestionable* form to describe the mathematical knowledge[8].

We propose the following reformulation of this "educational problem" into an explicit research problem within the ATD:

> **Problem of school mathematics connection:** How to design didactic praxeologies to articulate the mathematical curriculum both between domains and sectors in an educational level as between different levels? Particularly, what should be the characteristics of a didactic praxeology to take old contents up again, including those of previous school levels, and question, develop and integrate them into wider and more complex mathematical praxeologies?

Like any other problem of didactic research, it presents two inseparable faces:

- It is a *problem of mathematical engineering*, dealing with the analysis of mathematical praxeologies in the current curriculum and also with the *construction* of mathematical praxeologies. With relation to the *analysis*, we will ask about the nature of the limitations and insufficiencies of those mathematical praxeologies to engender and give sense to wider and more complex praxeologies, overcoming the *thematic level*. With relation to the *construction*, we will ask how to complete the existing praxeologies and how to connect them.
- It is a *problem of didactic engineering*, dealing with the construction of didactic praxeologies that allow the reconstruction of *wide* and *complex mathematical praxeologies*, overcoming the "theme

[6] Often, this responsibility is left in hands of the textbooks.

[7] Generally the interest is in innovative and motivational aspects. The lack of an epistemological question normally leads to the creation of isolated modelling processes and applications. They are supposed to be interesting on their own. There is no intention to build a wide and well-articulated set of modelling examples because supposedly it is not the mathematical knowledge contained in these situations that is problematic, but the way to transpose it into the classroom.

[8] In the formulation and solution of this educational problem, it is usual that "the mathematical" is identified with the "student's mathematical knowledge" (the knowledge they have to achieve) and "the didactic" with the classroom processes and mainly with the teacher's actions (see Gascón (1999) for a detailed explanation of the *educational problems* and the *didactic problems*). Again the teacher-student binomial, without taking into account the third didactic system component: the mathematical knowledge.

confinement" and connecting mathematical contents in each educational level and throughout different levels.

In the framework of the ATD numerous researches which can be connected to the general didactic problem of school mathematics articulation have been carried out[9]. In the following part of the paper we will focus on the teaching of *functional relationships* in Spanish Compulsory Secondary Education, following the researches of García (2005) and García and Ruiz (2006).

5. The process of studying functional relationships between magnitudes

The research problem treated in García (2005) starts from the study of the teaching and learning of *proportionality between magnitudes* in Spanish Compulsory Secondary Education. A first "spontaneous analysis[10]" of the current curricular documents and textbooks presently used shows a strong atomization in the current study processes developed in Spanish classrooms. This first *empirical fact* allows us to state that *the problem of the teaching and learning of proportionality* can be reformulated, from the research in didactics, as a manifestation of the *phenomenon* of the *disconnection of school mathematics*.

The *problem of the teaching and learning of proportionality* has been amply studied in the *cognitive programme* of research in mathematics education (see, for instance, Harel & Behr (1989), Hart (1988), Karplus, Pulos & Stage (1981, 1983a, 1983b), Lamon (1991), Noelting (1980a, 1980b), Singer & Resnick (1992), Tourniaire (1986), Tourniaire & Pulos (1985)). Although the theoretical frameworks and the methodologies used are varied, an isolation of *proportional relationships* from other kinds of relationships between magnitudes can be observed. In fact, many of these researches focus on the study of the "proportional reasoning" and on the fact that many students apply a proportional reasoning to non-proportional situations.

The *epistemological programme* depersonalizes the didactic question and considers the institutional mathematical activity as its primary object of study. It is assumed, as a main hypothesis, that not only "the mathematic" is dense in "the didactic", but also that, reciprocally, every *mathematical activity* is a *didactic activity* or a *study of mathematics* activity. The didactic problematic is widened including both the mathematical knowledge and the educational system (Chevallard, Bosch and Gascón, 1997).

We propose the reformulation of the *educational problem* of teaching and learning of proportionality in terms of the *didactic problem* of the connection of the functional relationships studied in Compulsory Secondary Education, starting from previous works of Bosch (1994), Bolea (2002) and García & Ruiz (2002).

5.1. The problem of connecting functional relationships between magnitudes in Compulsory Secondary Education

The research problem we are facing can be described as follows:

> How to design didactic praxeologies to connect the different relationships between magnitudes proposed by the curriculum, both between domains and sectors in a specific educational level and between different levels? What should be the characteristics of a didactic praxeology to take old contents up again, the ones related to the study of *variation systems between magnitudes*, including those of previous school levels, in order to question, develop and integrate them into wider and more complex mathematical praxeologies?

[9] Gascón (2001b) faces the disconnection between *synthetic geometry* and *analytic geometry* from Spanish Compulsory Secondary Education (students from 12 to 16 years old) to post-compulsory Secondary Education (16-18 years old). Bolea (2002), Bolea, Bosch and Gascón (2001a, 2001b, 2003) study the teaching of *algebra* as a modelling process. Fonseca (2004) and Bosch, Fonseca, Gascón (2004) analyses the changes in the study of mathematics in the transition between post-compulsory Secondary Education and University.

[10] We interpret a "spontaneous analysis" of the curricular documents and textbooks as that kind of analysis that only observes and describes the contents included in these documents and their distribution, without theoretical tools. This analysis does not pretend to be explicative but descriptive. It can be interpreted as a first contact with the *empirical field* that will be object of research later.

Moreover, we postulate that the rationale of the "relationships between magnitudes" in Secondary Education has its origin in the *problem of modelling systems* where two or more magnitudes can be considered as depending one on the others. The explicit construction of an *epistemological model* of the variation between magnitudes will be necessary. It will be the researcher's reference both to observe the actual mathematical praxeologies proposed to be reconstructed at school and to construct a new didactic organization allowing the development of a modelling process (in the ATD sense previously introduced) and generating a set of connected and integrated praxeologies around the variation systems.

5.2. An epistemological model of reference around the modelling of variation systems between magnitudes

In a general sense, in every *mathematical teaching system* we can find a *dominant epistemological model*, generally implicit, that is assumed by the individuals of that institution. This *model* has a crucial didactic relevance, because it determines how the "learning and teaching of mathematics" is conceived in that institution.

The *epistemological programme* postulates the necessity to explicitly construct an *epistemological reference model* (ERM from now on) of the mathematical knowledge involved. This model has to be provisional and open to further modifications according to the obtained results. This ERM will also be essential to study the *mathematical knowledge* before it is transformed to be taught (*didactic transposition processes*).

In García (2005) we constructed an ERM that starts form the questioning and characterization of different types of variations between magnitudes. In order to do that, and taking into account the institutional restrictions coming from Compulsory Secondary Education, we have opted to consider only the relationship between two magnitudes M and M' and also to consider that the set of magnitude quantities is a discrete one ($\{a_1, a_2, ..., a_i, ...\}$ represents a set of quantities of the magnitude M and $\{a'_1, a'_2, ..., a'_i, ...\}$ the correspondent quantities in M').

We will also consider that the starting point is a set of quantities of M in arithmetic progression ($k \in M$ is the difference between two consecutive terms) and we will question the nature of the variation in the corresponding quantities in M'. We have introduced five *variation types* (or *variation conditions*) that we summarize as follows[11]:

- *Equity condition*: every arithmetic progression $\{a_i\}$ of M elements and difference k is transformed into an arithmetic progression of M' elements and difference k'.

$$\forall k \in M \ , \ \dot{\exists}\ k' \in M' \ /$$
$$\Delta a_i = a_{i+1} - a_i = k \Rightarrow \Delta a'_i = a'_{i+1} - a'_i = k'.$$

- *Linear condition*: more restrictive than the previous one, it implies that not only every arithmetic progression of M quantities is transformed into an arithmetic progression of M' quantities, but also that every geometric progression of M quantities is transformed into a geometric progression of M' quantities with the same ratio.[12]

$$\forall k \in \Re \ ,$$
$$\nabla a_i = \frac{a_{i+1}}{a_i} = k \ \Rightarrow \ \nabla a'_i = \frac{a'_{i+1}}{a'_i} = k$$

  This condition can also be formulated in continuous terms: if $(a, a') \in M \times M'$ is a system *state* (that is, a pair of related quantities), then (ka, ka') and $(k^{-1}a, k^{-1}a')$ are also system *states* $(\forall k \in \Re - \{0\})$.

- *n-level difference constant condition*: every arithmetic progression $\{a_i\}$ of M elements and difference k is transformed into a progression with the n-level differences constant and equal to k'.

$$\forall k \in M \ , \ \dot{\exists}\ k' \in M'$$
$$\Delta a_i = k \ \Rightarrow \ \Delta^n a'_i = \Delta^{n-1} a'_{i+1} - \Delta^{n-1} a'_i = k'$$

- *Constant ratio condition*: every arithmetic progression $\{a_i\}$ of M elements and difference k is transformed into a geometric progression with a ratio k'.

$$\forall k \in M \ , \ \dot{\exists}\ k' \in \Re$$

[11] A wider description can be found in García (2005)
[12] The use of the notation ∇a_i has no relation with its proper use in mathematics as the gradient vector.

$$\Delta a_i = k \quad \Rightarrow \quad \nabla a'_i = \frac{a'_{i+1}}{a'_i} = k'$$

- *"Inverse linear" condition*[13]: every geometric progression of *M* quantities and ratio *k* is transformed into a geometric progression of *M'* quantities and ratio 1/*k*.

$$\forall k \in \Re,$$

$$\nabla a_i = \frac{a_{i+1}}{a_i} = k \;\Rightarrow\; \nabla a'_i = \frac{a'_{i+1}}{a'_i} = \frac{1}{k}$$

In continuous terms, if $(a, a') \in M \times M'$ is a system *state* then $\left(ka, \frac{1}{k}a'\right)$ is also a system *state* $(\forall k \in \Re - \{0\})$

From this point of view, the *direct proportional relationship* is reformulated as a relation between two magnitudes characterized by the *linear condition*. Equally, the *inverse proportional relationship* is the relation characterized by the *"inverse linear" condition*. However new kinds of relationships (as quadratic and exponential) can be considered depending on the *variation condition* assumed.

The ERM starts from the integrated study of systems where quantities of magnitude can vary under different *variation conditions*, as the ones described above or others. It then "evolves" with the construction, extension and integration of different kinds of variation praxeologies and generates a regional mathematical organization articulated by the theory of *real-valued functions*.

In García (2005) we have used this ERM as a tool to characterize the praxeologies proposed to be studied in the textbook usually used in the current Spanish Compulsory Secondary Education. Now we will illustrate how it can be used to elaborate a *study process* as an answer to the *curricular problem of the connection of functional relationships*.

5.3. A modelling process dealing with the variation systems: "The savings plans"

[13] We have chosen this name using the analogy with the "linear condition" although the term "inverse linear" can be contradictory from a strict mathematical point of view.

Every *study and research activity* starts from a *productive question Q* that allows the emergence of a kind of problems and a technique to solve them, as well as a technology to justify and understand the mathematical activity performed (Chevallard, 1999).

If this *productive question Q* is fertile enough, it can give rise to new problematic questions that will generate new types of tasks to be solved, producing a sequence of articulated mathematical praxeologies in a relatively large period of time, that is, a *study and research course* (SRC from now on).

The question of the rationale that motivated the creation and development of a mathematical content and that justified its inclusion in the syllabus of a teaching institution should be formulated at the beginning of every study process. In the case of "real functions", although a precise formulation of a unique *productive question* is not possible, it is evident that the origin of that question is related to the study of *variations*. Such question should deal with the characterization of *type of variation*. Provisionally, we will enunciate the *productive initial question* as follows:

> Q_i: Starting from a *situation* S_i where it is possible to establish a relationship between magnitudes, some of them varying in relation to one other, what are the characteristics of that variation?

That leads us to a more general *productive question* that we will use as the starting point for the generation of the SRC:

> Q_{var}: what principles can be used to define different "types of variation"?

We conceive the term *system* as a praxeology or, at least, as a set of praxeological components including a minimum of two magnitudes and a *technological* component that gives sense to a possible relationship between them. The extra or intra mathematical nature of this technological component, that determines the context where the system is placed, will be considered as a secondary aspect in the mathematical activity that will arise from the study of the system variation (although it is necessary to choose this context carefully if we want to ensure the *cultural* and *social legitimacy* of the constructed praxeology).

In the SRC that we have designed, we propose to place the system in an *economic-commercial context* (building of *savings plans*) because it is a familiar context to students in Secondary Education and also a *part of society* that *school* should take into account if it really wants to be an instrument to "improve citizens' lives[14]". Moreover, being coherent with the *epistemological reference model* constructed before, this context gives sense to discrete sets of quantities for each magnitude (for example, considering time divided in months, weeks, days, etc.).

Once the first step has been done (*level* 0: system delimitation and placement), it is time to delimit the magnitudes involved and to formulate the *productive question* that launches us into the *study process* (*level* 1, that has been assumed by the researcher, although other decisions could be taken):

- V_1 → the "savings plan" (*SP* from now on) period of time and the temporal distribution of the instalments.
- V_2 → the amount of money accumulated in each instalment.

We will also suppose that there is a one-to-one relationship between V_1 and V_2. That is, for a specific *SP*, each instalment (V_1 quantity) is associated to only one amount of saved money (V_2 quantity). Taking into account that the study process starts from the different characterizations in the variation of these magnitudes, there is a third variable in play: the evolution of the amount of money given in each instalment. Obviously, this variable is not independent of the other two. Indeed, as a quantification of V_1 to V_2 variation, this variable really represents the different values of the derivative of this function.

[14] Chevallard (2001, 2006) considers that the current dominating *school epistemology* is characterized by the elimination of the *raison d'être* (rationale) of the praxeologies proposed to be studied at school. A *monumentalization phenomenon* of these praxeologies, which are taken to school as ready-made objects (valuable in themselves) intended for students to visit, is produced. It is vital to modify the *school epistemology* to make a place for the *raison d'être* of the studied praxeologies if we pretend compulsory education to be a means to "improve citizens' lives".

The *delimitation* just explained is enough to formulate a *productive question* that will be able to produce/start up the desired SRC and, in particular, to transfer to students the necessity of a *second construction level*. Taking into account the previous *construction levels* 0 and 1, this crucial question ("savings question") will be the following:

Q_S: what principles can be used to plan a specific "savings plan" (SP_i)?

We consider that it really is a *productive question* because: (A) its generality makes new structuring and delimitation levels necessary (but now as part of students' responsibility), (B) it is able to produce a mathematical activity starting from a relatively simple *mathematical context* (elementary arithmetic techniques), and (C) it allows the construction and simulation of different savings plans, which will emerge firstly as *specific praxeologies*.

The *study community task* (but mainly the student's task) will be:

- *To choose a first system state*, that is, a starting amount of money. This first state is provisional and could be changed. It acts as a parameter of the situation.
- *To decide how the next states will be generated*, that is, the *variation condition* that characterizes the system. There is not a unique way to accomplish this task. It requires choosing between two system variables.

  Firstly, it is necessary to decide on a temporal distribution that will rule the saving. A way to do that is through an equal distribution of the instalments (for instance, daily, weekly, monthly, etc.). In that case, the set of V_1 measures can be identified with the discrete set $\{0,1,2,3,...\}$. Obviously, other instalments distribution could be considered (even an arbitrary distribution or a continuous one).

  Secondly, it is necessary to decide the amount of money to save in each instalment. Again, there are too many possibilities, from an arbitrary distribution (a different amount in each instalment, not related with the previous) to a distribution following a rule. However, we consider a *recurrence rule* appropriate (taking into account the institutional restrictions and the mathematical activity that we want to develop). That is: *if I give an amount C_n in the instalment n, I have to give an amount C_{n+1} in the instalment n+1 related to C_n in the same form as C_n was related to C_{n-1}.*

- *To simulate the system*, that is, to construct a set of states wide enough to develop the experimental activity needed in the study process.

Although students have a lot of freedom, it is reasonable to expect the emergence of savings plans with equal temporal distribution of instalments and recurrent rules. We propose the following (although it is possible to consider different ones).

Equitable variation savings plans (Eq): a fixed amount C is given in each instalment.

Accumulative with increasing amount savings plans (Var_{Ac}): a higher amount is given in each instalment. It can be distinguished depending on the evolution of that amount:

- Var_{Ac}^1: if an amount C is given in the first instalment then the same amount given in the previous one plus C is given in each instalment (if we give C in the first instalment, we will give $C + C$ in the second one, $2C+C$ in the third one and so on).
- Var_{Ac}^2: if an amount C is given in the first instalment then a multiple (by a constant factor $k > 1$) of the previous one is given in each instalment (if we give C in the first instalment, we will give $k \cdot C$ in the second one, $k^2 \cdot C$ in the third one and so on). For instance, if we consider an increase of 15%, then $k = 1{,}15$.

Accumulative with amount share savings plans: a lower amount is given in each instalment. Depending on the evolution of that amount, we can distinguish:

- Var_{Ac}^3: if an amount C is given in the first instalment then the same amount given in the previous one minus a "discount amount D" is given in each instalment (if we give C in the first instalment, we will give $C - D$ in the second one, $(C - D) - D$ in the third one and so on).
- Var_{Ac}^4: if an amount C is given in the first instalment then a multiple (by a constant factor $0 < k < 1$) of the previous one is given in each instalment (if we give C in the first instalment, we will give $k \cdot C$ in the second one, $k^2 \cdot C$ in the third one and so on). For instance, if we consider a decrease of 15%, then $k = 0{,}85$.

5.4. The development of the "savings plans" study and research course

The study and research course will start from the simulation of different savings plans depending on the variation condition chosen. First of all, students have to choose the number of instalments, the temporal distribution and the initial parameters (initial amount and, if needed, the auxiliary quantities to calculate the evolution of this amount). The simulation will lead them to a final saved amount C_f.

For the management of the *moment of the first encounter*, the following task can be proposed to our students (as a particularization of the previous *productive question*):

> ***Q***$_S$: "*We want to plan the final course trip with enough time. We have to decide different ways to save money to achieve the amount of money needed for this trip. Although we do not know this quantity yet, we can start making an estimation of money needed and taking decisions about our personal savings plans: number of instalments, shares, etc. Obviously, it is not our task to decide today how much money we have to give and how, but try to anticipate the necessities we will have when we know the expenses of the trip by the end of the year*".

This initial question opens a great variety of decisions to be taken: the number of instalments and their temporal distribution, the existence of an initial amount (C_0) and the evolution of the amount given across the savings plan. It is reasonable to foresee (and it really happened in our experimentations) that students first consider *equitable* savings plans.

Anyway, once a variation type has been decided (not necessarily the same for all students), a first task emerges "naturally": testing of our savings plans (calculate the money accumulated in each instalment) and observing what happens[15]. We have called this first type of tasks "simulation tasks" ($T_{simulation}$), which provokes the transition to and the development of the *exploratory moment*.

The performance of this type of task leads to the creation of different primitive arithmetic techniques (that we call $\tau_{arithmetic}$) depending on the type of variation. It is possible to introduce a spreadsheet to accomplish this task, giving rise to a "mixed technique" (variation of the SRC that we used in the two experimentations carried out). For instance, measuring instalments in months and supposing that students have chosen an *accumulative with increasing amount savings plan* Var^1_{Ac}, this *arithmetic* technique can be described as follows:

τ^1_{Ac}: according to Var^1_{Ac}, in each instalment we have to give the same amount as in the previous one but increased by C:

x (months)	0	1	2	3	4	5	
y (euros)	C_0	y_1	y_2	y_3	y_4	y_5	

$+C$ $+(C+C)$ $+(2C+C)$

Obviously, it is possible to construct such techniques for each type of variation. The construction of these techniques gives rise to a set of *specific praxeologies* constructed for each type of variation: *SP*(*Eq*), *SP*(Var^1_{Ac}), *SP*(Var^2_{Ac}), *SP*(Var^3_{Ac}) and *SP*(Var^4_{Ac}).

The mathematical activity that is possible to develop with these *specific praxeologies* is rather limited. That limitation is more evident when we do not only want to calculate system states but also to control the system (in the sense of taking decisions concerning the parameters), that is, when we want to know the appropriate initial quantities to anticipate the system development and, mainly, to construct systems in accordance with the desired final amount C_f. We have called these types of tasks *control tasks*[16]. The previous techniques also show their weakness to accomplish *comparison tasks* among different savings plans.

The importance of this type of tasks is that the previous *arithmetic techniques* learned by students work successfully in some of them but fail in others. This problem is more evident in non-equitable savings plans and provokes the necessity to widen the initial *specific praxeologies* with the aim of getting techniques that will ensure the control and anticipation of the system (first, the students choose the initial quantities at random and act by essay-error but they soon notice the difficulties to get to the desired final amount). We have called this type of tasks *algebraic modelling tasks* (T_{alg_mod}).

The mathematical activity gives rise to the creation of different *local praxeologies* that we name $MO_L(Eq)$, $MO_L(Var^1_{Ac})$, $MO_L(Var^2_{Ac})$, $MO_L(Var^3_{Ac})$ and $MO_L(Var^4_{Ac})$, depending on the variation type. The *study community* has to work on each savings plan type with the aim of constructing algebraic models that connect the initial quantities with the final amounts. In the development of our SRC, this moment corresponds to the *moment of working on the technique*. The formulation of the system variation as a recurrent rule makes possible to obtain an algebraic model working on the different states (*technique of recurrence*: τ_{rec}): *starting from the amount saved in an instalment n* (y_n), *work on it trying to decompose it in terms of the previous amounts until the initial quantities are reached* (parameters of the system).

For instance, if we are working on an *accumulative with increasing amount savings plan* (Var^1_{Ac} type):

$$\tau_{rec}: \quad \begin{aligned} y_0 &= C_0 \\ y_1 &= C_0 + C \\ y_2 &= y_1 + 2C = C_0 + C + 2C \\ y_3 &= y_2 + 3C = C_0 + C + 2C + 3C \end{aligned}$$

[15] Sometimes we reach a great amount in few instalments but, at other times, we only obtain a little amount after a lot of instalments. The feedback provided by the system will provoke a variation of the initial quantities and the performance of new simulations.

[16] Working on a specific savings plan and given a final amount C_f, we can formulate at least three different *control tasks* (in the simplest case) depending on the known quantities and the one to be calculated (playing with three variables: the number n of instalments, the first instalment C and its evolution and the initial amount C_0).

. . . .

$$y_n = y_{n-1} + nC = C_0 + \sum_{k=1}^{n} k \cdot C$$

The general formula $y_n = C_0 + \frac{n(n+1)}{2} C$ can be obtained (given that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$). In this formula, time is considered as a discrete magnitude. Making a variable change ($n \rightarrow x$), extending this variable to the set of real numbers and working on the expression, it is possible to obtain the *algebraic model*:

$$y = f(x) = \frac{C}{2} x^2 + \frac{C}{2} x + C_0$$

This *algebraic model* allows the construction of different but connected techniques to control the evolution of the *SP* and also the comparison between different *SP*. When an *algebraic model* is constructed for each type of variation, it is also possible to compare savings plans of different nature. It then emerges a connected and integrated mathematical activity around different types of variation.

In a general way, the development of the experimented SRC is summarized in figure 1. The different tasks and techniques will emerge several times during the study process and depend on the variation type considered. García (2005) provides a more detailed description of each task and each technique.

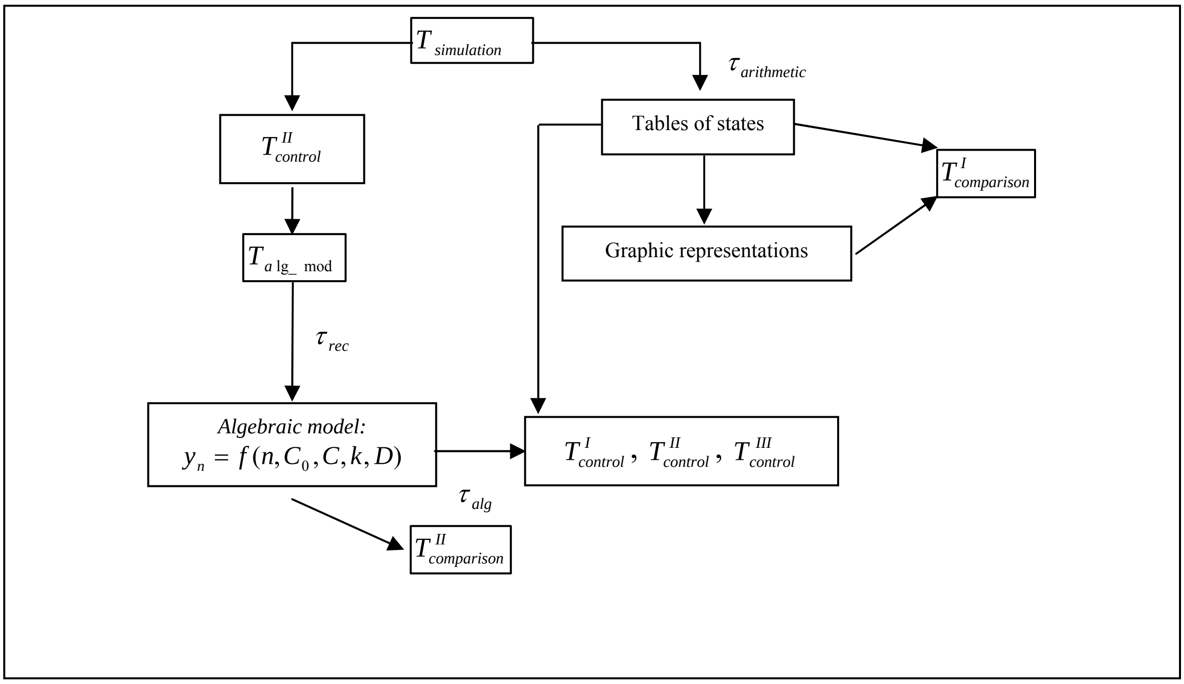

Figure 1. Study and Research Course "The Savings Plans"

Thus, working on each type of variation, *controlling* and *comparing* different savings plans, it is possible to get a set of praxeologies of increasing complexity (summarized in figure 2) giving rise to a *regional praxeology* $MO_R(SP)$ articulated by the *theory of real-valued functions*, that is, a *modelling process* on the study of functional relationships.

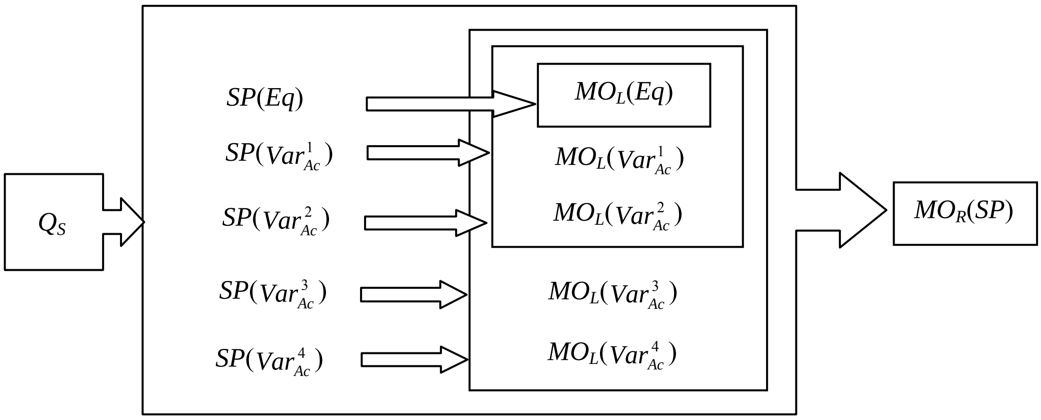

Figure 2. Articulation, integration and amplification of praxeologies in the "Savings Plans" SRC

To finish with our description, it is important to stop for an instant in the *moments of institutionalization and evaluation*. Traditionally, these *moments* correspond to the instant when the work done is taken up again, reviewed, reorganized, tested and put in relation to our previous knowledge. Usually, the *dominant school epistemology* assigns this role to the teacher. We have tried to delegate this responsibility to the students through the request of a report for the school principal that would be useful for their school partners in the following years.

6. Summary and conclusions

We started this paper with a revision of the research done in mathematics education in the "modelling and applications" domain. As a consequence of this analysis, we propose a reformulation of the notion of *modelling*, both to identify new didactic phenomena and to formulate new research problems a well as possible ways to face them.

One of these didactic phenomena, directly related to the lack of mathematical modelling processes at school, is the *disconnection of school mathematics*, which extends to almost all *mathematics teaching system* levels. This *disconnection* is provoked by many factors. The identification of those factors is a wide and difficult research problem. However, the Anthropological Theory of the Didactic (ATD) provides a valuable didactic tool to go deeper in its origins and to elaborate possible solutions at least to reduce the effects of this phenomenon.

Although it is true that official curricular documents consider the problem solving activity, in general, and the mathematical modelling, in particular, as didactic tools useful to integrate curricular contents (see, for instance, the Andalusia curriculum or the standard "connections" included in the Standards of the National Council of Teachers of Mathematics), it seems difficult for the integration to be produced *by itself*. It is important to take into account that the way the research interprets mathematical modelling (that essentially depends on the theoretical position and framework assumed/adopted by the researcher) determines the role it could play in the school mathematics integration.

In our research we proposed to reformulate the modelling processes as processes of reconstruction and connection of praxeologies of increasing complexity (*specific* → *local* → *regional*) that should emerge from the questioning of the rationale of the praxeologies that are to be reconstructed and connected.

We focused on the problem of *the connection of functional relationships between magnitudes in Compulsory Secondary Education* (in Spain). After the precise identification and formulation of this problem in the ATD, we designed and carried out a *study process*: "the savings plans", that constitutes not only a modelling example, but as a research result.

To elaborate this study process as a *study and research course*, we have had to explicitly construct an *epistemological reference model* of the variation systems between magnitudes. The *study and research course* designed has played a double function in our research: it has shown the power and relevance of the reformulation of the modelling notion as a didactic analysis tool and it provides a solution to the problem of *the connection of functional relationships between magnitudes.*

Although we already carried out two implementations of the *study and research course* in two school in Andalusia during 2004 and 2005 (with students from 14 to 16 years old, see García, 2005), we consider that new experimentations are necessary to go deeper in the characterizing of these modelling processes that imply the construction, amplification and integration of a set of praxeologies. It would be especially crucial to progress in the knowing of the set of cultural, didactic and mathematical restrictions that hinder the development of these modelling processes at school.

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