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13 a 15 de Novembro de 2011

GeoGebra e o método de Briot & Bouquet para resolução gráfica de equações cúbicas.

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Novembro de 2011

Charles Auguste Briot, matemático francês, nasceu no dia 19 de julho de 1817 em St. Hippolyte. Foi responsável por importantes contribuições em análise, calor, luz e eletricidade. Apesar de ter perdido o movimento do braço, devido a um acidente de infância, nunca desistiu de ser um professor.

Em 1838, um ano após sua chegada em Paris, começou a estudar na *Ecole Normale Supérieure* (1838), onde obteve um doutorado (1842) defendendo um trabalho sobre a órbita de um corpo sólido ao redor de um ponto fixo. Tornou-se professor no *Orléans Lycée* e posteriormente na Universidade de Lyon, onde reencontrou o amigo de infância *Claude Bouquet*, com quem fez um trabalho importante em análise.

Em 1851, voltou para Paris, onde ensinou em vários liceus e foi professor substituto em diversos cursos superiores. Ensinou cálculo, mecânica e astronomia, especialmente na *Ecole Polytechnique* e na *Faculté des Sciences*. Na década seguinte, em 1864, tornou-se professor da *Sorbonne* e da *Ecole Normale Supérieure*. Briot escreveu muitos livros importantes na área de ensino, e recebeu várias honrarias pelo seu trabalho. Morreu no dia 20 de setembro de 1882, em Bourg-d'Ault, França.

(1817 – 1882)

Jean Claude Bouquet foi amigo de escola de Briot e trabalhou com ele durante grande parte de sua vida. Bouquet entrou na École Normale Supérieure, em 1839, obtendo seu doutorado em 1842 com uma tese sobre a variação de integrais duplas. Foi nomeado professor de matemática no Liceu de Marselha, em seguida, ele foi para Lyon, como professor de matemática na Faculdade de Ciências. Em Lyon, ele reuniu-se novamente com seu amigo de escola, Briot e os dois começaram uma parceria que durou ao longo de suas carreiras.

De 1852 até 1858 Bouquet ensinou no Liceu Bonaparte (mais tarde renomeada no Lycée Condorcet). Em 1858 mudou-se para o Liceu Louis-le-Grand, o Galois escola havia se formado a partir de 30 anos antes. Ele lecionou até 1867. Bouquet Em 1874 foi nomeado professor de cálculo diferencial e integral na Sorbonne. Ele ensinou lá até 1884, um ano antes de sua morte.

(1819 – 1885)

BRIOT AND BOUQUET'S

ELEMENTS

OF

ANALYTICAL GEOMETRY

OF TWO DIMENSIONS

The Fourteenth Edition

TRANSLATED AND EDITED

BY

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INSTRUCTOR IN MATHEMATICS IN THE UNIVERSITY OF CHICAGO

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Moyennant le versement de ce droit d'auteur vous pouvez considérer comme seule autorisée la traduction que vous ferez de l'ouvrage en langue anglaise.

Veuillez agréer, monsieur, l'assurance de mes sentiments les plus distingués,

C. Delagrave

(*Translation.*)

(DEAR SIR:

As agreed to by both parties your translation will be regarded as the only authorized translation of this work (Briot et Bouquet Geometrie Analytique) in the English language.

Accept, dear sir, my highest compliment,

C. DELAGRAVE.)

Book IV *

THE GENERAL THEORY OF CURVES

CHAPTER I

The Construction of Curves in Rectilinear Co-ordinates.

CHAPTER II

Convexity and Concavity.

CHAPTER III

Asymptotes.

CHAPTER IV

Construction of Curves in Polar Co-ordinates.

CHAPTER V

Concerning Similitude.

CHAPTER VI

Graphic Solution of Equations.

CHAPTER VII

Notions concerning Unicursal Curves.

CHAPTER VI*

GRAPHIC SOLUTION OF EQUATIONS.

402. Consider two equations in two unknown quantities x and y

(1) $\phi(x, y) = 0,$ (2) $\psi(x, y) = 0;$

each of them defines a curve. For this system of two equations can be substituted an infinity of equivalent systems of equations; consider in particular a system

(3) $\chi(x, y) = 0,$ (4) $f(x) = 0,$

one of which only contains the variable y, a system which may be obtained by eliminating y from the two given equations. The real roots of equation (4) are the abscissas of the points common to the two curves (1) and (2). And yet, if the system of equations (3) and (4) were satisfied by a pair of values of the form $x = \alpha$, $y = \beta + \gamma i$, in which α, β, γ are real, these values would satisfy the system of equations (1) and (2); but the quantity α would not be the abscissa of a real point common to the two curves. The exception which we have pointed out never occurs when the equation $\chi(x, y) = 0$ is an algebraic equation which involves only y to the first degree.

When one wishes to solve an equation $f(x) = 0$ in a single unknown quantity, the curves determined by (1) and (2) may be selected in an infinity of different ways. The only condition to be fulfilled is that the elimination of y between equations (1) and (2) give the proposed equation. A first combination is $y = f(x)$, $y = 0$, which leads one to consider the values of the unknown quantity as the abscissas of the points of intersection of the curve $y = f(x)$ with the x-axis. This combination is rarely the most simple. It is proven in algebra that

if an unknown quantity y be eliminated from two algebraic equations in two unknown quantities whose degrees are m and n, the resulting equation in x is at most of the degree mn. Consequently, if the proposed equation be algebraic and one wish to obtain its roots by the intersection of algebraic curves, the product of the degree of the equation of the two curves would be equal to the degree of the equation to be solved. We shall apply this method to the solution of the equation of the fourth degree.

403. The equation of the fourth degree may be easily reduced to the form

(5) $$x^4 + px^2 + qx + r = 0;$$

it may be regarded as the result of the elimination of y between the two equations of the second degree

(6) $x^2 - my = 0,$ (7) $m^2y^2 + pmy + qx + r = 0,$

each of which defines a parabola. Since equation (6) involves y only to the first degree, all the real roots of equation (5) are the abscissas of real points common to the two curves.

One can substitute for the parabola (7) another curve of the second degree which passes through the intersection of the curves (6) and (7). The general equation of the second degree satisfying this condition (§ 277) is

(8) $$kx^2 + m^2y^2 + qx + m(p - k)y + r = 0,$$

k being an arbitrary parameter. If one put $k = m^2$, the curve (8) would be simply a circle; the co-ordinates a and b of the center and the radius R of this circle are given by the formulas

(9) $$a = -\frac{q}{2m^2},\ b = \frac{m^2 - p}{2m},\ R^2 = a^2 + b^2 - \frac{r}{m^2}.$$

When the value of R^2 is positive, equation (8) represents a real circle, and the real roots of equation (5) are the abscissas of the points of intersection of this circle and parabola (6). When the value of R^2 is negative, equation (8) cannot have real solutions (§ 85); the same is true of the system of equations

(6) and (7), or of the equivalent system of equations (5) and (6); the four roots of equation (5) will be imaginary.

404. Consider next the equation of the third degree reduced to the form

$$x^3 + px + q = 0.$$

If this equation be multiplied by x, which introduces the root $x = 0$, an equation of the fourth degree is obtained,

$$x^4 + px^2 + qx = 0,$$

to which the preceding method is applicable. The value of R^2, being in this case equal to $a^2 + b^2$, is always positive. The circle and the parabola pass through the origin of co-ordinates; the abscissa of this point is the root $x = 0$, which one should remove.

The same parabola $x^2 - my = 0$ may serve for the solution of all equations of the third or of the fourth degree; the circle only changes depending upon the value of the coefficients of the proposed equations. This method can be employed with advantage when one would solve successively a great number of equations; then a parabola having an arbitrary parameter is traced with great care; and, in each particular example, it only remains to determine the circle.

405. When the unknown quantity x is a line, and the unit of length has not been specified, the equation $f(x) = 0$ is a homogeneous equation in the unknown quantity x and the various known lines. In case the equation is of the fourth degree, if the coefficients p, q, r be rational functions, or irrational functions of the second degree of given lengths, on taking an arbitrary length for the parameter m of the parabola, the co-ordinates of the center and the radius of the circle could be constructed with the rule and the compass.

But if the equation be a numerical equation, that is if the coefficients be given numbers, a definite value is given to m; for example, one would put $m = 1$, and construct the parabola and the circle by means of an arbitrary scale; the abscissa of

one of the points of intersection measured by the same scale, will give the value of the unknown quantity x.

We know that the solution of two equations of the second degree in two unknown quantities x and y, or the determination of the points of intersection of two curves of the second degree, reduces to the solution of an equation of the fourth degree in one unknown quantity. This solution could therefore be accomplished by means of a definite parabola and a circle. Accordingly, if one of the curves of the second degree be already traced, it can be used with the circle.

406. EXAMPLE I.—Draw through a given point P whose co-ordinates are x_1 and y_1 a normal to a parabola $y^2 - 2px = 0$. The co-ordinates x and y of the foot of the normal are determined by the system of equations

$$y^2 - 2px = 0, \quad xy - (x_1 - p)y - py_1 = 0.$$

If all the terms of the last equation be multiplied by y, and if $2px$ be substituted for y^2, a new parabola $x^2 - (x_1 - p)x - \frac{y_1 y}{2} = 0$ is obtained; on adding the equations of the two parabolas member to member, one obtains the circle $x^2 + y^2 - (x_1 + p)x - \frac{y_1 y}{2} = 0$. The points where this circle intersects the given parabola are the feet of the normals (§ 306).

407. EXAMPLE II.—*Solve the numerical equation* $x^3 - x - 7 = 0$. Construct by means of an accurately made scale, the parabola $x^2 = y$; describe a circle whose center C has the co-ordinates $a = \frac{7}{2}$, $b = 1$, and which passes through the origin; this circle intersects the parabola in one additional point M; therefore the proposed equation has but one real root, the abscissa OP of the point M (Fig. 261). By measuring this length by means of the scale here employed, we find $x = 2.09$.

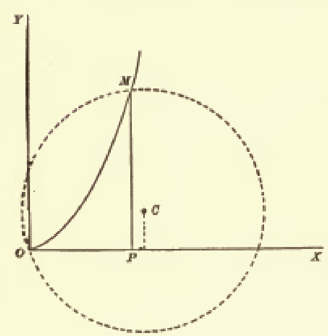


Fig. 261.

EXAMPLE III.—Solve the equation $x^3 - 5x + 1 = 0$. Construct the circle whose center C has the co-ordinates $a = -\frac{1}{2}$, $b = 3$ and which passes through the origin: this circle intersects the parabola in three points; it follows that the equation has three real roots; on measuring the abscissas, one finds that the two positive roots are 0.20 and 2.13.

408. Example IV. — Consider the transcendental equation

$$x \operatorname{tang} x = 1.$$

This equation is the result of the elimination of y between the two equations

$$y = \operatorname{tang} x, \quad xy = 1.$$

The first represents a curve composed of an infinitude of equal branches which have asymptotes perpendicular to the x-axis; the second an equilateral hyperbola (Fig. 262). It is evident that the right-hand branch of

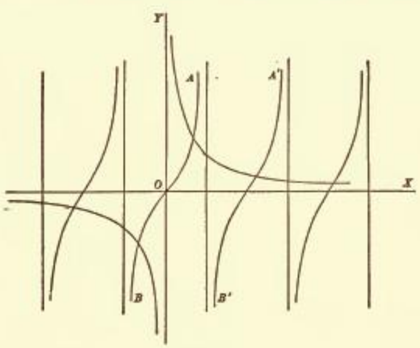


Fig. 262.

the hyperbola intersects, at least once, each of the branches OA, $B'A'$, ... of the transcendental curve; moreover there is but a single point of intersection on each branch, because, when x varies, the ordinates of the two curves vary in contrary directions; if these ordinates be equal for a certain value of x, they are necessarily unequal for every other value. The roots of the equation are equal in pairs with contrary signs; there is in the first place a root situated between 0 and $\frac{\pi}{2}$, a second root between π and $\frac{3\pi}{2}$, a third between 2π and $\frac{5\pi}{2}$, etc., ...; the number of roots is infinite. On calling x_n the nth root, the difference between x and $(n-1)\pi$ is very small when n is very large. The curve gives, for the value of the first root, 0.86.

The equations $y = \operatorname{tang}\left(\frac{\pi}{2} - x\right)$, $y = x$ could also be discussed in this manner on putting $\frac{\pi}{2} - x = x'$, $y = \operatorname{tang} x'$, $y = \frac{\pi}{2} - x'$; the hyperbola would be replaced by a straight line.

409. Remarks. — The graphic methods which we have described do not give the values of the unknown quantities with any great precision; one should not expect an approximation nearer than the hundredth part of the root.

One often attacks the problem by magnifying the traces of the two curves in order to determine the number of real roots of an equation. But, so long as the form of the two curves is not studied with care, no rigorous conclusions can be deduced from this discussion. In general, the discussion of the curves and the determination of the points of intersection offer the same difficulties as the problem proposed.

407. Example II. — *Solve the numerical equation* $x^3 - x - 7 = 0$.

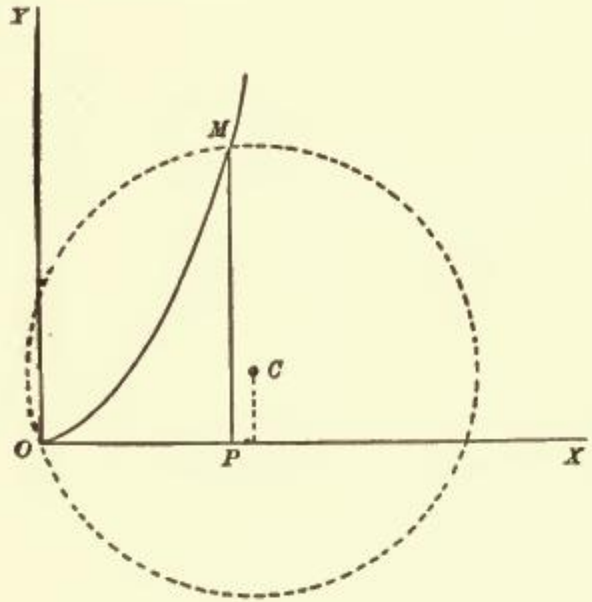


Fig. 261.

Construct by means of an accurately made scale, the parabola $x^2 = y$; describe a circle whose center C has the co-ordinates $a = \frac{7}{2}$, $b = 1$, and which passes through the origin; this circle intersects the parabola in one additional point M; therefore the proposed equation has but one real root, the abscissa OP of the point M (Fig. 261). By measuring this length by means of the scale here employed, we find $x = 2.09$.

Example III. — Solve the equation $x^3 - 5x + 1 = 0$. Construct the circle whose center C has the co-ordinates $a = -\frac{1}{2}$, $b = 3$ and which passes through the origin: this circle intersects the parabola in three points; it follows that the equation

Considere a equação da circunferência que passa pela origem do sistema cartesiano e tem centro no ponto C (m, n), isto é, $(x-m)^2+(y-n)^2=m^2+n^2$ e uma mudança de variável dada por $y=x^2$, que consiste na transformação de uma equação de circunferência em um sistema dado pela circunferência e pela parábola com vértice na origem.

Assim os pontos que satisfizerem a equação transformada também satisfazem o sistema.

$$(x-m)^2+(y-n)^2=m^2+n^2 \Leftrightarrow (x-m)^2+(y-n)^2-m^2-n^2=0$$
$$=x^2-2mx+\cancel{m^2}+y^2-2ny+\cancel{n^2}-\cancel{m^2}-\cancel{n^2}=0$$

Assim tem-se que $x^2-2mx+y^2-2ny=0 \xleftrightarrow{y=x^2} x^2-2mx+x^4-2nx^2=0$.

Reagrupando tem-se que: $x^4-(2n-1)x^2-2mx=0 \Leftrightarrow x\left(x^3-(2n-1)x-2m\right)=0$

Essa ultima equação possui quatro raízes, sendo 3 referentes a equação de terceiro grau $x^3+px+q=0$ e $x=0$, onde $p=-2n+1 \Leftrightarrow 1-p=2n \Leftrightarrow n=\frac{1-p}{2}$ e $q=-2m \Leftrightarrow m=-\frac{q}{2}$.

Desta forma pode-se proceder da seguinte forma:

À equação $x^3+px+q=0$ acrescenta-se a raiz $x=0$, e obtêm $x^4+px^2+qx=0$.

Assim tem-s e o seguinte sistema: $\begin{cases} x^4+px^2+qx=0 \\ y=x^2 \end{cases}$.

Fazendo a substituição da 2ª equação na 1ª vê-se que:

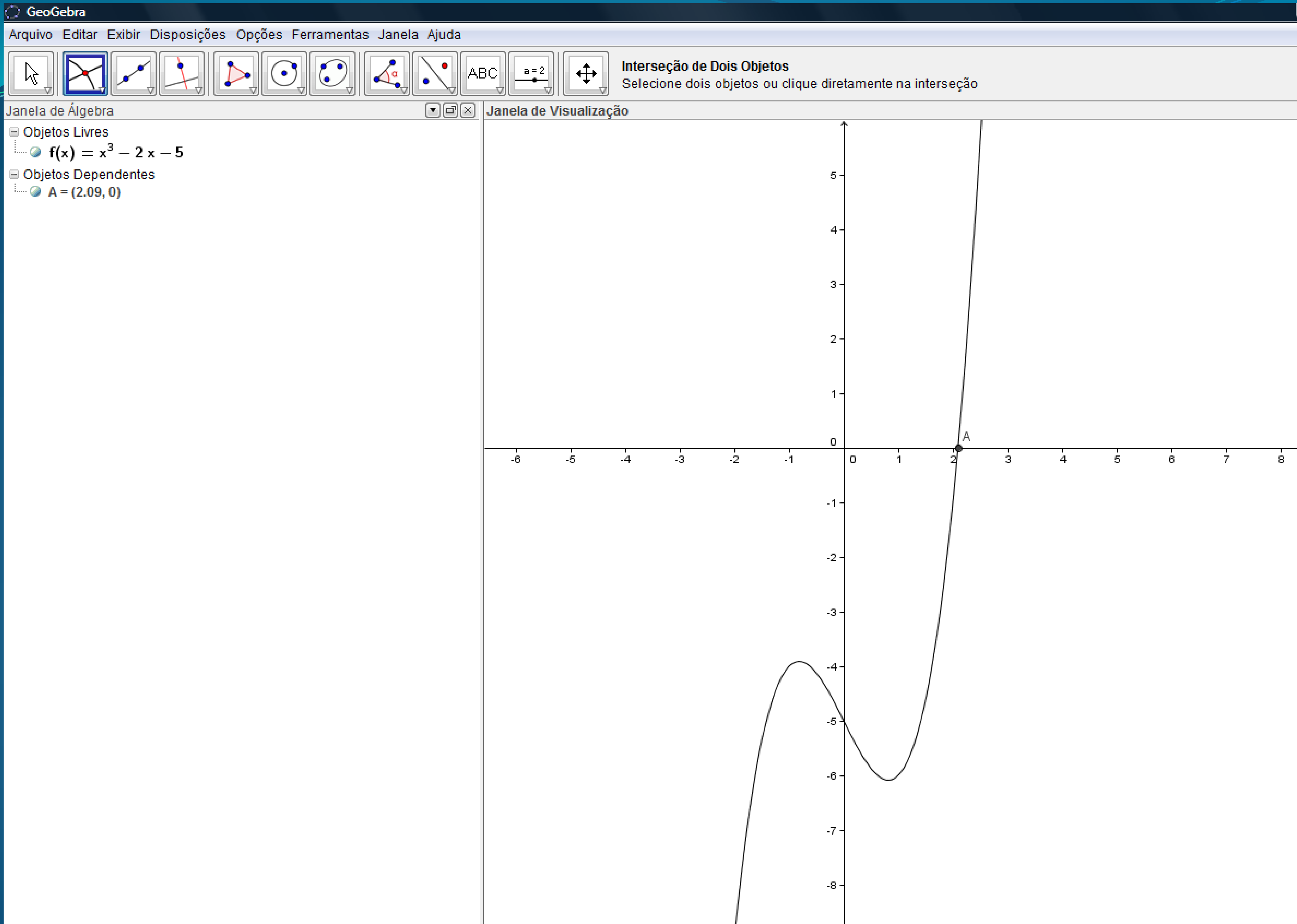
$$x^4+px^2+qx=0 \xleftrightarrow{y=x^2} y^2+py+qx=0$$

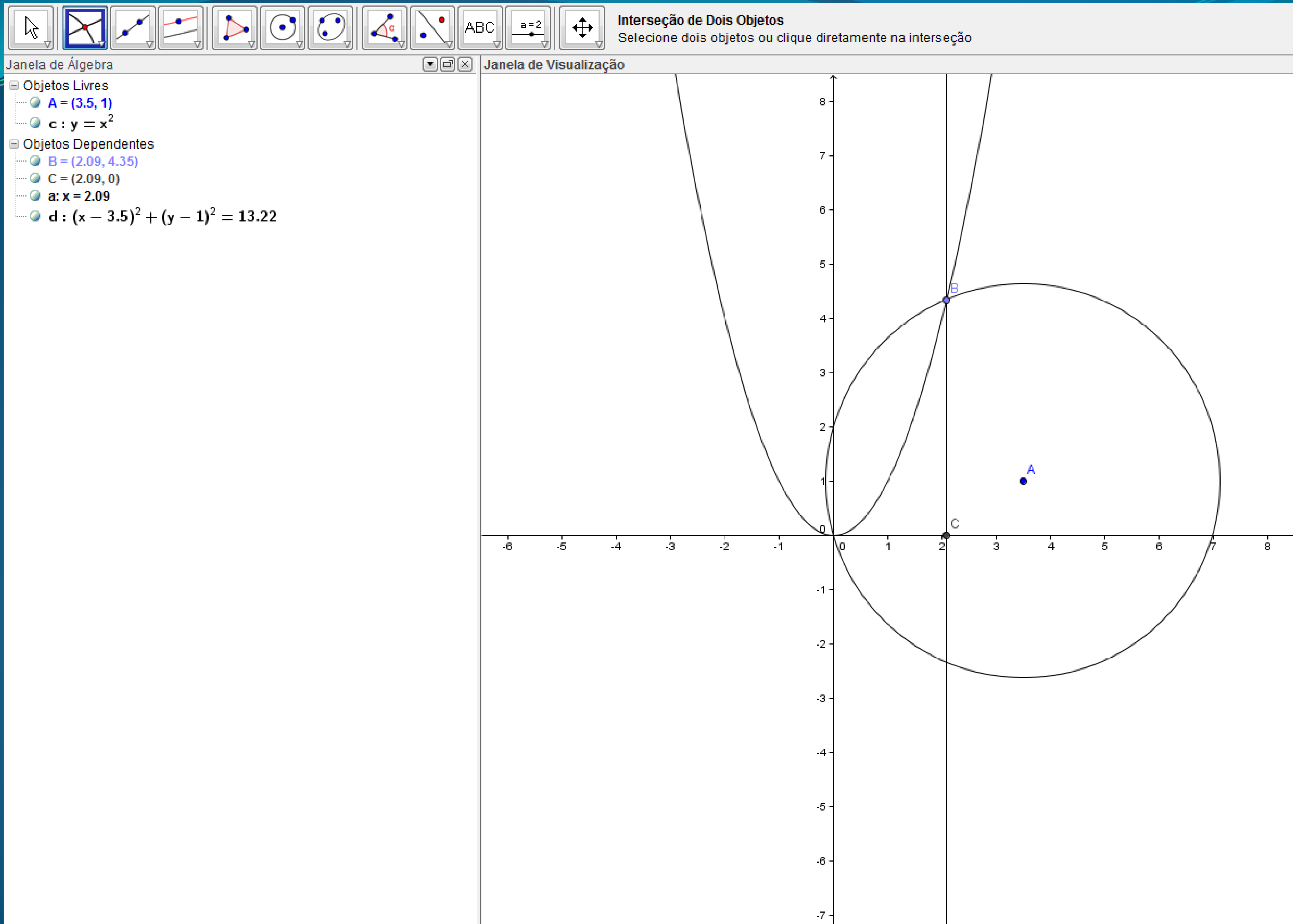
Completando os quadrados para as duas variáveis obtêm-se:

$$\left(y-\frac{p}{2}\right)^2-\frac{p^2}{4}+\left(x+\frac{q}{2}\right)^2-x^2-\frac{q^2}{4}=0 \Leftrightarrow$$
$$\Leftrightarrow \left(x+\frac{q}{2}\right)^2+\left(y+\frac{p}{2}\right)^2-x^2-\frac{\left(p^2+q^2\right)}{4}=0 \xleftrightarrow{y=x^2}$$
$$\Leftrightarrow \left(x+\frac{q}{2}\right)^2+\left(y+\frac{p}{2}\right)^2-y-\frac{\left(p^2+q^2\right)}{4}=0 \xleftrightarrow{m=-q/2}$$
$$\Leftrightarrow \left(x-m\right)^2+\underbrace{y^2+py+\frac{p^2}{4}-y}_{y^2+(p-1)y+\frac{p^2}{4}}-\frac{\left(p^2+q^2\right)}{4}=0 \xleftrightarrow{n=1-p/2}$$
$$\Leftrightarrow \left(x-m\right)^2+\left(y-n\right)^2-\frac{p^2}{4}-\underbrace{\frac{q^2}{4}}_{m^2}-\frac{1}{4}+\frac{2p}{4}=0 \Leftrightarrow$$
$$\Leftrightarrow \left(x-m\right)^2+\left(y-n\right)^2-m^2-\frac{\left(p-1\right)^2}{4}=0 \Leftrightarrow$$
$$\Leftrightarrow \left(x-m\right)^2+\left(y-n\right)^2-m^2-\left(-n\right)^2=0 \Leftrightarrow$$
$$\Leftrightarrow \left(x-m\right)^2+\left(y-n\right)^2=m^2+n^2, \text{ com } m=-\frac{q}{2} \text{ e } n=\frac{1-p}{2}.$$

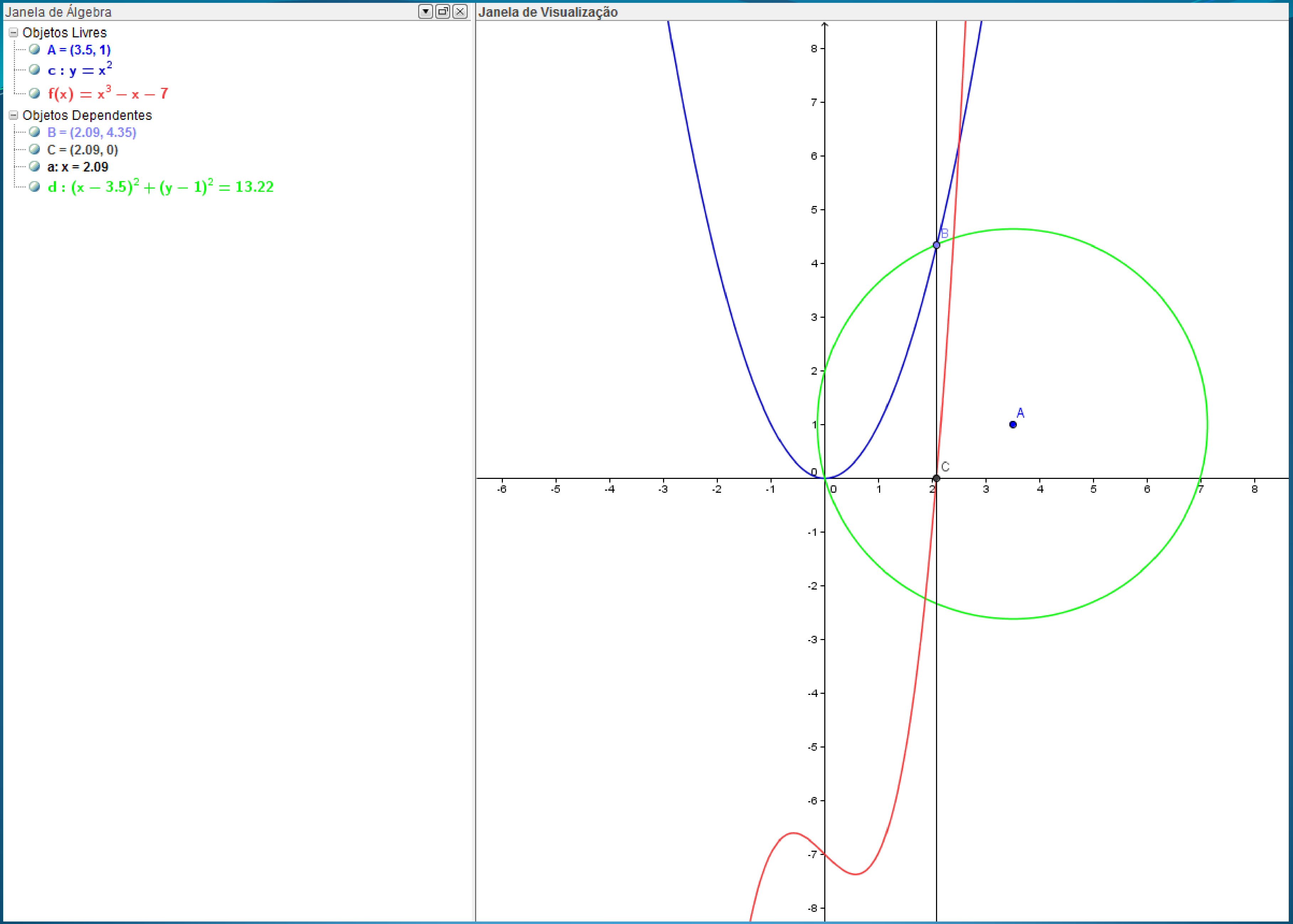
Desta maneira, a interpretação geométrica da álgebra descrita é que as 1ªs coordenadas dos pontos de interseção entre a circunferência que passa na origem do sistema e tem centro no ponto C (m, n) e a parábola $y=x^2$ são as quatro raízes da equação $x^4+px^2+qx=0$.

Tirando a raiz acrescentada $x=0$, têm-se as três raízes da equação $x^3+px+q=0$.

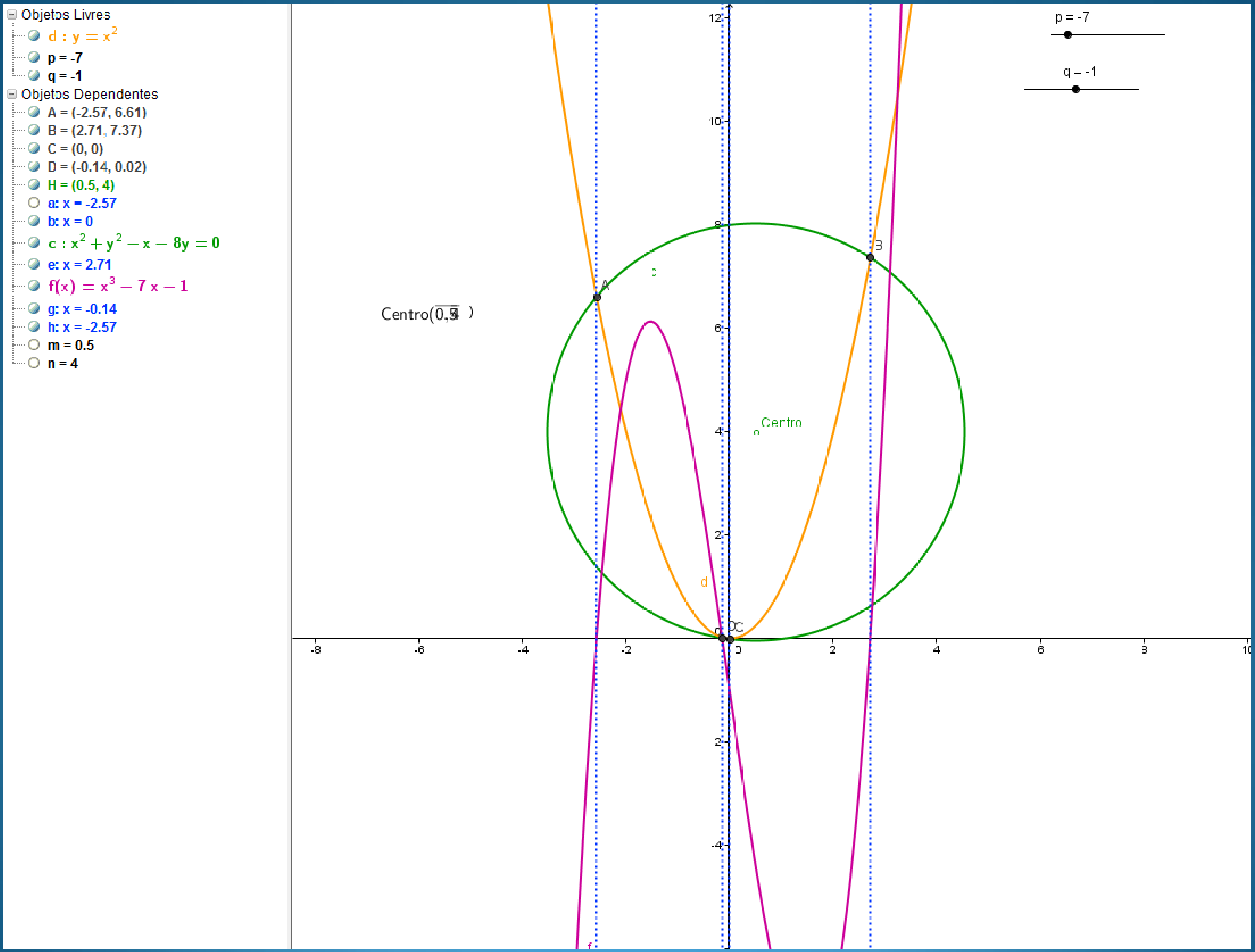

GeoGebra
Arquivo Editar Exibir Disposições Opções Ferramentas Janela Ajuda
Interseção de Dois Objetos
Selecione dois objetos ou clique diretamente na interseção
Janela de Álgebra
Objetos Livres
f(x) = x³ − 2 x − 5
Objetos Dependentes
A = (2.09, 0)
Janela de Visualização
A



Interseção de Dois Objetos
Selecione dois objetos ou clique diretamente na interseção
Janela de Álgebra
Objetos Livres
A = (3.5, 1)
c : y = x²
Objetos Dependentes
B = (2.09, 4.35)
C = (2.09, 0)
a: x = 2.09
d : (x − 3.5)² + (y − 1)² = 13.22
Janela de Visualização
B
A
C


Janela de Álgebra
Objetos Livres
A = (3.5, 1)
c : y = x²
f(x) = x³ − x − 7
Objetos Dependentes
B = (2.09, 4.35)
C = (2.09, 0)
a: x = 2.09
d : (x − 3.5)² + (y − 1)² = 13.22
Janela de Visualização
A
B
C

Implementação do Método de Briot & Bouquet



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